BRL

REPORT NO. 1128
APRIL 1961

NUTATION DAMPER - A SIMPLE TWO-BODY GYROSCOPIC SYSTEM

S. J. Zaroodny
J. W. Bradley

Department of the Army Project No. 503-03-001
Ordnance Management Structure Code No. 5010.11.814

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

ASTIA AVAILABILITY NOTICE

Qualified requestors may obtain copies of this report from ASTIA.

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

REPORT NO. 1128

APRIL 1961

NUTATION DAMPER - A SIMPLE TWO-BODY GYROSCOPIC SYSTEM

S. J. Zaroodny

J. W. Bradley

Exterior Ballistics Laboratory

Department of the Army Project No. 503-03-001
Ordnance Management Structure Code No. 5010.11.814

A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

REPORT NO. 1128

SJZaroodny/JWBradley/bjk
Aberdeen Proving Ground, Md.
April 1961

NUTATION DAMPER - A SIMPLE TWO-BODY GYROSCOPIC SYSTEM

ABSTRACT

Two rigid bodies are pinned together on an axis which is a principal axis of inertia for each body; they are free to spin about this axis, except for some friction between them; the system is free in space; one of these bodies is inertially asymmetric about this mutual axis; and - for simplicity - the other body is symmetric. If this assembly is inertially more nearly like a disc than a spindle, it settles in such a position that the mutual axis is aligned with the stationary vector of the angular momentum of this system. Thus it constitutes a means of internal and passive damping of the random nutations.

The behavior of this system is inspected by considering the numerable solution of an example (in which one of the two bodies is a thin disc, and the other is a thin rod which is always aligned with some diameter of the disc). The nature of the solutions is indicated sufficiently clearly by isometric views of the trajectories of the vector of angular momentum of the system in a coordinate system fixed with respect to the asymmetric body (these trajectories are termed, somewhat arbitrarily, "polhodes"). The two extremes of the friction

(the "clamped" and the frictionless assembly) are outlined, and it is shown that in a variation of the friction between these two extremes there occurs a rather peculiar change in the behavior of the system: a form of nutation corresponding to the "unstable" spin of a rigid body becomes "metastable". The emphasis of the presentation is on the legitimacy and usefulness of a rather primitive "empirical" mathematical approach.

TABLE OF CONTENTS

1. INTRODUCTION

This paper describes a nonlinear system of four ordinary first-order differential equations which represent the behavior of a rather basic assembly of two rigid bodies-a device which seems to be of particular interest because by virtue of its simplicity it may be ranked next to the classical problem of the nutations of a rigid body.*

This device has been suggested (by Dr. C. J. Cohen of Naval Weapons Laboratory, Dahlgren, Virginia, and others), as a "nutation damper" for observation satellites. It constitutes a passive system (one containing friction, but involving no power sources, no loss of mass of the satellite, and no external forces - such as magnetic) which converts the random nutations of the satellite into a smooth spin about the vector of the angular momentum of the system. As will be discussed, such a system can stabilize an assembly which - inertially - is roughly of an oblate shape. An ancient (though apparently never specifically formulated) desideratum of ballistics is an "internal stabilization" of a spindle. However, it appears - so far - that a prolate-shaped assembly can only be "destabilized" by the system here considered.

---

* In their review of nonlinear mechanics (1), Leimanis and Minorsky define as a basic class of nonlinear problems that of the motion of a rigid body fixed at a point. Basically, these are third-order problems (e.g., a gyroscope in gimballs is a three-body assembly, but has only three degrees of freedom). To that class one may add an important sub-class, the theory of projectile stability in ballistics; for a projectile is usually considered as a rigid body, and - though the problem is often linearized - many complications arise from the mathematical nonlinearity of the forces acting upon the projectile (2). The problem is usually construed as one of the fourth-order, but of a particular, "epicyclic," subclass of that order, which reduces to the second order in complex variables (3). On the other hand, many other radical complications of the general ballistic problem arise in those cases where the projectile may no longer be considered a rigid body (as just one example of this class of problems we may mention a projectile containing liquid), and even in the case where the projectile may no longer be considered a body having inertial axial symmetry. For these reasons, that large extension of Leimanis-Minorsky's classification, <u>the motion of systems of bodies</u>, will eventually be of interest; and in such a class, the problem here considered seems indeed the simplest one.

Our approach to the problem described here has been "empirical", viz., rather primitive: it has been simply an inspection of the numerical solutions of this system with "sample" inputs. This approach is handicapped by imperfections of the available analog computing machinery; yet, a general understanding of the behavior of such a system does emerge from such inspection. In particular, there have been found some unexpected peculiarities in the behavior of this system; they seem to leave room for hope that an eventual "internal-passive" stabilization of a spindle might not be impossible.

2. STATEMENT OF THE PROBLEM

Two rigid bodies are pinned together on an axis which is a principal axis of inertia for each body; they are free to spin about this axis, except for some friction between them; the system is free in space; one of these bodies is inertially asymmetric about this mutual axis; and - for simplicity - the other body is symmetric.

The standard Lagrangian approach requires an introduction of variables of analysis that completely define, at each instant, not only the angular velocities of these two bodies, but also their position (viz., such coordinates as the Eulerian angles ϕ, θ, ψ). A considerable simplification, however, becomes possible in our force-free case, through approaching this problem via a generalization of Euler's dynamical equations. The analysis then yields only a part of the problem: the polhodes (the paths of the vectors of angular velocity in the system-fixed coordinates); but this is the principal part of the problem. Given the polhodes as a function of time, the position of these two bodies at each instant can be - if and when needed - computed through the Euler-Poinsot interpretation of the motion as a rolling of the polhode cones on the herpolhode cones (the paths of the vectors of angular velocity in a Newtonian coordinate system); i.e., by a subsequent quadrature. This is particularly convenient in our force-free case, where the vector of the angular momentum of the system is constant in the Newtonian coordinate system.

3. DERIVATION OF THE EQUATIONS

Let the right-handed triads of the principal axes of the two bodies be 1, 2, 3 and 4, 5, 6, with 3 and 6 being the common axis (Figure 1). Let the moments of inertia be $I_1 \neq I_2$, I_3 for one body; and $I_4 = I_5$ (= I), I_6 for the other. The components of the angular velocities of the first body are ω_1, ω_2, ω_3; and of the second body, ω_4, ω_5, ω_6. Because of the constraint on axes 3 and 6 the same vector in the 1-2-4-5 plane is described by $\omega_{4,5}$ as by $\omega_{1,2,}$; thus the angular velocity of the second body can also be described by $\omega_{1,2,6}$.* Therefore, the behavior of this system can be described by a four-dimensional vector, viz., by the angular velocities ω_1, ω_2, ω_3, ω_6. Let the friction torque acting along the axis 3 on the first body be F; this is some odd function of the relative angular velocity of the two bodies

$$\beta \equiv \omega_6 - \omega_3$$

For simplicity we shall here assume the "viscous" friction $F = f\beta$, f a positive constant. Let also $I_1 + I \equiv K_1$, $I_2 + I \equiv K_2$, $I_3 + I_6 \equiv K_3$.

The components of the angular momentum $\underline{L}$ of the system on the moving axes 1, 2, 3 are related to the angular velocities by the following equations (the first two of which have the indicated simple form because of the simplifying assumption $I_4 = I_5$):

$$L_1 = (I_1 + I)\,\omega_1 = K_1\omega_1$$

$$L_2 = (I_2 + I)\,\omega_2 = K_2\omega_2$$

$$L_3 = I_3\omega_3 + I_6\omega_6$$

* However, this vector suffices to define the angular momentum of the second body only in the cases of the axial symmetry of this body, when any pair (1,2) of transverse axes are the principal axes.

Then the equations of motion are

$$\dot{L}_1 - L_2\omega_3 + L_3\omega_2 = 0$$
$$\dot{L}_2 - L_3\omega_1 + L_1\omega_3 = 0$$
$$\dot{L}_3 - L_1\omega_2 + L_2\omega_1 = 0$$
$$I_6\dot{\omega}_6 = - F$$

The first three equations express the conservation of the vector of the angular momentum $\underline{L}$ in space; they are simply a statement that the derivative of this vector, which in our nutating system 1, 2, 3 is $\dot{\underline{L}} - \underline{L} \times \underline{\omega}$, is zero. In this nutating system this vector is conserved only in magnitude, and its direction continually changes. The fourth equation is simply Euler's equation for the axis 6, and its simplicity is another result of the assumption $I_4 = I_5$. These equations can be re-written in such forms as

$$\text{(a)} \quad \begin{cases} K_1\dot{\omega}_1 = \omega_2 \left[(K_2 - I_3)\,\omega_3 - I_6\omega_6\right] \\ K_2\dot{\omega}_2 = \omega_1 \left[(I_3 - K_1)\,\omega_3 + I_6\omega_6\right] \\ I_3\dot{\omega}_3 = \omega_1\omega_2 \,(K_1 - K_2) + F \\ I_6\dot{\omega}_6 = - F \\ F = (\omega_6 - \omega_3)\, f \end{cases}$$

$$\text{(b)} \quad \begin{cases} \dot{\omega}_1 = \omega_2 \left[\dfrac{K_2 - K_3}{K_1}\,\omega_3 - \dfrac{I_6}{K_1}\,\beta\right] \\ \dot{\omega}_2 = \omega_1 \left[\dfrac{K_3 - K_1}{K_2}\,\omega_3 + \dfrac{I_6}{K_2}\,\beta\right] \\ \dot{\omega}_3 = \omega_1\omega_2 \dfrac{K_1 - K_2}{I_3} + \dfrac{F}{I_3} \\ \dot{\beta} = - \omega_1\omega_2 \dfrac{K_1 - K_2}{I_3} - F\left(\dfrac{1}{I_3} + \dfrac{1}{I_6}\right) \\ F = f\beta \end{cases}$$

$$
(c) \quad \begin{cases} \dot{L}_1 = L_2 \quad \left[(I_3^{-1} - K_2^{-1}) \quad L_3 - I_3^{-1} L_6\right] \\ \dot{L}_2 = -L_1 \quad \left[(I_3^{-1} - K_1^{-1}) \quad L_3 - I_3^{-1} L_6\right] \\ \dot{L}_3 = L_1 L_2 \quad (K_2^{-1} - K_1^{-1}) \\ \dot{L}_6 = - F = - f\beta \\ \beta = (I_3^{-1} + I_6^{-1}) \; L_6 - I_3^{-1} L_3 \end{cases}
$$

Each one of these forms has some advantages. Thus, form (a) might appear the most natural; form (b) is particularly convenient when one needs to consider the mechanism of the friction in detail; and form (c) is particularly convenient in visualizing the problem, for three reasons: firstly, it allows a visualization of L_1, L_2, L_3 as coordinates of a point on a sphere*; secondly, the third equation is free of β; and thirdly, the "angles of yaw" of the axes 1, 2, 3 nutating about the stationary $\underline{L}$ are, of course, simply the angles from the axes 1, 2, 3 to the vector $\underline{L}$ in the nutating system (whose directional cosines are L_1/L, L_2/L, L_3/L). The term "polhode" applies, strictly, to the paths of the tips of the three-dimensional vectors $\omega_{1,2,3}$ and $\omega_{1,2,6}$, and perhaps ought not be applied to the four-dimensional vectors $\omega_{1,2,3,6}$ and $L_{1,2,3,6}$. Yet the paths of the latter vectors can readily be converted to the true polhodes, and do specify the behavior of the system sufficiently clearly for our purposes.

* This is particularly convenient in the work with an analog machine, since it facilitates keeping track of the imperfections of the machine.

4. FORMULATION OF THE NUMERICAL EXAMPLE

Our system is seen to be defined by no less than <u>ten</u> parameters. With our empirical approach (and for our descriptive purposes) this is entirely too many, and we need to simplify the problem drastically, yet attempting to lose as little of the generality as possible.

Five of these parameters characterize the system "inertially": in the forms (a) and (b) of our equations they are ratios such as I_6/K_1, $(K_2 - I_3)/K_1$, etc.; in the form (c), expressions such as $(1/I_3 + 1/K_2)$, etc. These five parameters are restricted somewhat by the fact that the I's and the K's are moments of inertia; these restrictions amount to the fact that the I's must be positive and such that they can form a triangle, viz.,

$$I_1 + I_2 > I_3 > |I_1 - I_2|, \text{ etc.}$$

It can be shown that the number of the independent inertial parameters can be reduced from 5 to 3. However, for our present purposes such complete systematization is not imperative, and we shall not attempt it here*. Rather, we shall limit ourselves to a single numerical example where

$I_6/I = 2$ (i.e., the second body is a "thin disc"), and

$I_1 = 0$, $I_2/I = I_3/I = 0.1$ (i.e., the first body is a small "thin rod" along the axis 1).

The sixth parameter is the friction parameter, f/I_6, f/I_3 or f in our three forms of the equations. It completes the specification of the system itself.

The remaining four parameters are the initial conditions; that is, (in our three forms of equations) the initial values of $\omega_{1,2,3,6}$, of $\omega_{1,2,3}$ and β, or of $L_{1,2,3,6}$.

---

* Some remarks on departure from these assumptions are given in section 12. For the important case f = 0 a more thorough systematization of the inertial parameters of this system has been made by Masaitis (6).

To simplify our arithmetic we may select more convenient units for moments of inertia, angular momentum, time and friction coefficient. Although distinctly non-linear, our system does possess a modicum of homeogeneity. E.g., in the form (c) the expressions for the derivatives with respect to time are linear and homogeneous in the reciprocals of the moments of inertia; hence we may use an arbitrary unit of the moments of inertia if we adjust the unit of time accordingly. Furthermore, the first three of these expressions are of the second order in the L's, and the fourth is linear in both L and f. As far as the second-order terms are concerned, a further change in the unit of time is equivalent to multiplying all initial conditions by a constant; and as far as the terms containing f are concerned, such change is equivalent to a change of the friction coefficient*. Thus we are free to take an arbitrary unit of time, provided we change the unit of f accordingly. We shall select the units of time, angular momenta, moments of inertia and f in such a way that

$$I = 1 \text{ and } L^2 = 1$$

We shall also restrict ourselves to the natural case $\beta(0) = 0$, which corresponds to the situation where the two bodies are originally clamped together, and the operation of our mechanism starts at the instant they are unclamped**.

---

* That is, groups of terms of the type $dL_1/dt = L_2L_3/I$ can be interpreted as $dL_1/d(t/a) = L_2L_3/(I/a)$, and further, as $d(bL_1)/d(t/b) = (bL_2)(bL_3)/I$, where a and b are arbitrary; while groups of the type $dL_6/dt = fL/I$ can be interpreted as $d(bL_6)/d(t/b) = (bf)(bL)/I$, etc. With such transformations the solutions based on the assumption $L^2 = I = 1$ can always be interpreted in the conventional units for L and I (the unit of time being ab; of I, a = I; and of L, 1/b).

** E.g., the two bodies forming a satellite are clamped by the acceleration of the propelling rocket, and the operation of the mechanism starts when the propulsion ceases.

5. CONSTANTS OF MOTION

Two physical quantities are of particular interest in our study.

One of these is the magnitude of the angular momentum of the system, $\underline{L}$, given by

$$L^2 = (K_1\omega_1)^2 + (K_2\omega_2)^2 + (I_3\omega_3 + I_6\omega_6)^2$$
$$= L_1^{\ 2} + L_2^{\ 2} + L_3^{\ 2}$$

Since there are no external forces acting on the system, this is a constant of motion*. This is one of the reasons for our preference of the form (c) of our equations.

The other quantity is the kinetic energy T of the system, given by

$$2T = K_1\omega_1^{\ 2} + K_2\omega_2^{\ 2} + I_3\omega_3^{\ 2} + I_6\omega_6^{\ 2}$$
$$= L_1^{\ 2}/K_1 + L_2^{\ 2}/K_2 + (L_3 - L_6)^2/I_3 + L_6^{\ 2}/I_6,$$

where $L_6 = I_6\omega_6$.

Generally, this is not a constant of motion; for it is readily seen (particularly from physical considerations) that T is in general dissipated in friction, at the rate which is the work consumed in friction. In fact, by multiplying the first four equations of (a) by ω_1, ω_2, ω_3, ω_6, and adding, we obtain

$$\dot{T} = -\beta F = -f\beta^2 \leq 0$$

* This can be readily checked by multiplying the first three equations of (c) by L_1, L_2, L_3, adding and integrating.

In fact, this is the principle of operation of this mechanism. If there is any energy flow because of the friction, the system settles in a state in which $\beta = 0$ and T is a minimum. Generally, one should expect this to occur about that axis about which the moment of inertia is a maximum*. It is our object to inspect whether such damping of the nutation actually occurs, and if so, to find how rapid it is, and how it can be expedited.

In two important extreme cases, however, T is indeed a constant of motion. The obvious one of these is the case of infinite friction, when the system degenerates into a single rigid body. The other is the more interesting (and perhaps novel) case of zero friction**.

Let us now inspect the effect of the parameter f.

* The underlying principle of mechanics is invoked often enough in specific applications, but - to our knowledge - has never been stated in the full force and generality it deserves; it seems akin to Schwartz's inequality and to the second law of thermodynamics. Cf. also (5).

** In this case, of course, any combination of T, L^2 and L_6 can be a constant of motion. However, only L^2 would survive as such when the friction is introduced.

6. VERY LARGE FRICTION

With f → oo it is readily seen that (if the torque $f\dot{\beta}$ is to retain its physical significance, i.e., remain finite) $\beta \to 0$, and the whole assembly becomes a rigid body with the moments of inertia K_1, K_2, and K_3; equations (c) become, accordingly

$$\dot{L}_1 = L_2 L_3 \, (K_3^{-1} - K_2^{-1})$$

$$\dot{L}_2 = L_3 L_1 \, (K_1^{-1} - K_3^{-1})$$

$$\dot{L}_3 = L_1 L_2 \, (K_2^{-1} - K_1^{-1})$$

The solutions of these equations are well known*, but will be reviewed briefly, in order to "blend" them with those of the more complicated cases of $0 < f < \infty$. The kinetic energy of the system is

$$T = L_1^{\,2}/2K_1 + L_2^{\,2}/2K_2 + L_3^{\,2}/2K_3$$

---

* The formal solutions are in terms of elliptic functions - which, in fact, can best be defined (4) as suitably-scaled solutions of these equations. These solutions can be summarized as follows: The separatrix between the two groups of loops of Figure 2 lies in a plane $L_3/L_1 = \sqrt{(K_3/K_1)(K_2 - K_1)/(K_3 - K_1)} = S$, say. For nutations about axis 1, and initial conditions in the 1-3 plane, the solutions are: $L_3 = L_{30}\,\mathrm{cn}(k,u)$; $L_2 = L_{2max}\,\mathrm{sn}(k,u)$; $L_1 = L_{10}\,\mathrm{dn}(k,u)$. The modulus $k = (L_{10}/L_{30})\,S$ is 0 at axis 1 and 1 on the separatrix. The independent variable is $u = t/U$, the unit of time being $U = K_1$ times $\sqrt{K_2K_3/(K_3 - K_2)(K_3 - K_1)}/L_{30}$. For nutations about axis 3 one may either consider $k > 1$, or (as is more customary) take $1/k$ as modulus, with $L_1 = L_{10}\,\mathrm{cn}\,(1/k,u)$, $L_3 = L_{30}\,\mathrm{dn}(1/k,u)$, etc. It should be emphasized that the modulus k of the elliptic functions characterizes not only the physical system, but also the initial conditions. This is somewhat as though a change in the initial conditions changed the "system"; which, after all, is the basic characteristic of non-linear systems.

and the tip of the vector $\underline{L}$ (in our nutating system 1, 2, 3) therefore lies on the surface of an ellipsoid with semiaxes $\sqrt{2K_1 T}$, $\sqrt{2K_2 T}$, $\sqrt{2K_3 T}$. The paths of $\underline{L}$ are the intersections of this ellipsoid with the sphere L^2 = const (= 1), and the motion therefore is periodic. Figure 2 shows these paths for our case of $K_1/K_2/K_3 = 1/1.1/2.1$, which is an example of a disc-shaped, or "oblate", assembly; in this case the nutations about axes 1* and 3 are stable, while the spin about 2 is unstable. The T-ellipsoid always has the same proportions, but - it is very important to note - the magnitude of T must be between two extremes, for these intersections to occur. In our case T is maximum for a pure spin about 1, when the ellipsoid is wholly outside the sphere, touching it only at its smallest semiaxis $\sqrt{2K_1 T}$; and it is minimum for a pure spin about 3, when the ellipsoid is wholly inside the sphere, touching it only at its largest semiaxis $\sqrt{2K_3 T}$.

Incidentally, from the condition $\dot{\beta} = 0$ in the equations (b) it is readily seen that the torque F necessary to hold these two bodies "frozen" is

$$F = -\omega_1\omega_2 (K_1 - K_2)/(1 + I_3/I_6),$$

so that the inequality $K_1 \neq K_2$ (or $I_1 \neq I_2$), or (as J. Sternberg put it) an "incompatibility" of these two bodies, is a necessary condition for invoking the mechanism of this friction.

* In a Newtonian coordinate system, of course, it is the axis 1 which nutates about the stationary $\underline{L}$; etc.

7. FAIRLY LARGE FRICTION

With f finite but sufficiently large we should expect that the motion of the two-body system would be somewhat like that of a rigid body, and the trajectories of $\underline{L}$ (or $\underline{\omega}$) would be similar to those of Figure 2; except that these trajectories will keep "slipping" towards the loops of the lower kinetic energy, viz., generally from axis 1 to axis 3. The assembly thus becomes a "nutation damper", ending always with the pure spin about axis 3. This is illustrated, for the case f = 10, in Figure 3 which has been drawn by the differential analyzer.

Four features distinguish Figure 3 from Figure 2:

First, the multiplicity of the trajectories of Figure 2 is replaced, in effect, by a single long trajectory (or by any trajectory intermediate between any two swings of this long trajectory). Of course, a motion starting in a close proximity to axis 1 would be difficult to obtain on a differential analyzer, for such motion would be slow in starting (it might best be obtained by running the trajectory backwards).

Second, the question whether the motion settles on the positive or the negative axis 3 is a question of whether the passage from what resembles one set of loops on Figure 2 to the other occurs after "bouncing away" from one or the other "saddle point"; for the reader may sense the existence of two such saddle points, roughly in the vicinity of the positive and the negative axis 2. Thus, when starting with small nutations about axis 1, the whole assembly can settle with axis 3 either parallel, or anti-parallel, to the space-fixed vector $\underline{L}$. Figures 3 and 4 (and one trajectory on Figure 5) show motion ending at the "north pole" of the sphere. On each figure the intersections of the long trajectory with the meridian passing through axis 1 thus mark a series of points starting at which the motion goes to the north pole. The same would hold for sufficiently closely neighboring points on this meridian; so this series of points becomes a series of sub-arcs of this meridian. In the region of loops about the pole these sub-arcs blend together; but in the region of loops about axis 1 the situation is more complicated. It can easily be "interpolated" (and with the differential analyzer it can easily be shown "experimentally") that for some trajectories which start between such sub-arcs in this region the motion would "bounce away" from a saddle point near the negative axis 2, and would proceed to the south pole. Therefore in this region this meridian can be divided into a series of alternate sub-arcs (and the whole region, into a series of alternate stripes), starting in which the motion would go either to the north or to the south pole. On the rest of the sphere the situation is obviously symmetrical, the symmetry being one of a rotation through 180° about axis 2. It is natural to inquire what would happen to those two trajectories which are the precise boundaries between these two "stripes"; one might expect that they would fall into one or the other saddle point on positive or negative axis 2 and would "stay there". In the following (in section 10) we shall touch upon this matter briefly; but here we need only warn the reader that there are complications: the saddle points are not necessarily on axis 2, and are not necessarily points of (even unstable) equilibrium. However, it is obvious that any trajectory in a close vicinity to these two "critical" trajectories may spend an extremely long time in a close vicinity of one or the other saddle point near axis 2.

Third, the "undamping" of the nutations about 1 proceeds faster than the damping of the nutations about 3. This feature of our mechanism, however, may be affected by a change in the law of friction employed.

Finally, our three-dimensional representation of a four-dimensional problem is necessarily crude. Properly, the representation should include some statement on the fourth variable (L_6, ω_6 or β), whose variation corresponds to any parameter distinguishing the different trajectories on Figure 2*.

* Such curves are easily obtained on the differential analyzer, but are omitted here for simplicity. The variation of any such fourth variable has the character of rapid oscillations superposed upon a slow change; but it is, perhaps, only an indication of this slow change that is of interest as a parameter varying along our long trajectory. The rapid oscillations, of course, are of interest as indicating the mechanism by which the "incompatibility" of the two bodies invokes the friction.

8. MODERATE FRICTION

One should expect that between the two extremes of friction, $f = \infty$ and $f = 0$, there might exist a particular value of friction at which the damping of the nutations is "maximized"; and that this is to be sought for friction $f < 10$. Indeed, Figures 4 and 5, representing the cases of $f = 2$ and $f = .5$, do show an increase in the rate of damping. Two features, however, may be noted. The gain in the rate of this damping diminishes, so that this rate is apparently approaching some maximum; and the "undamping" of the nutations about 1 becomes much more pronounced and faster than the damping of the nutations about 3.

In fact, the passage from $f = \infty$ to $f = 0$ is accompanied by a rather puzzling, almost "qualitative", change in the behavior of the system: there arises the tendency for $\underline{L}$ to stick for a long time in the vicinity of axis 2*. We shall presently see that this complexity can be traced to the fact that of the three pairs of equilibrium points of the single-body system (the points on the axes 1, 2, 3 in Figure 2), only one remains truly an equilibrium point when the friction is introduced. This is the pair on axis 3 ($L_1 = L_2 = \beta = 0$). The other two cease to be equilibrium points.

This phenomenon might be best inspected by considering next the other (and very interesting) extreme, the case of $f = 0$.

* This tendency seems to be quite distinct from (though it is undoubtedly related to) the fact that the two "critical" trajectories on Figures 3, 4, 5 would arrive at some "saddle points" and "stay there".

9. ZERO FRICTION

With $f = 0$, L_6 (or ω_6) becomes a constant parameter, and our fourth-order system degenerates into another third-order system:

$$\dot{L}_1 = L_2 \left[(I_3^{-1} - K_2^{-1}) L_3 - I_3^{-1} L_6 \right]$$

$$\dot{L}_2 = - L_1 \left[(I_3^{-1} - K_1^{-1}) L_3 - I_3^{-1} L_6 \right]$$

$$\dot{L}_3 = L_1 L_2 \quad (K_2^{-1} - K_1^{-1})$$

It may be noted that with $L_6 = 0$ these are, in effect, the equations for the nutations of a rather hypothetical rigid body having the moments of inertia K_1, K_2 and I_3. In our example $I_3 \ll K_1 < K_2$, and this hypothetical body is a thin spindle along axis 3, which is considerably different from our device made rigid. The solutions, of course, are the elliptic functions cn, sn, dn of time.

On the other hand, the terms in L_6, if they could be taken by themselves (e.g., in the case of a very large L_6), could be viewed as forming a system

$$\dot{L}_1 = - L_2 (L_6 I_3^{-1})$$

$$\dot{L}_2 = + L_1 (L_6 I_3^{-1})$$

which represents the circular nutations of another hypothetical, axially-symmetric (and again considerably different from our physical system) rigid body with the moments of inertia $K_1 = K_2 =$ any K, and $K_3 = I_3 + K$; the solutions being the circular functions cos, sin - plus a constant (for L_6). Thus the functions L_1, L_2, L_3 for this zero-friction case might be said to have the following relation to the circular and elliptic functions:

If the system $\dot{\underline{x}} = \underline{f}(\underline{x})$ yields elliptic functions, and the system $\dot{\underline{y}} = \underline{g}(\underline{y})$ yields circular functions, the functions L_1, L_2, L_3 are given by the system

$$\dot{\underline{L}} = \underline{f}(\underline{L}) + \underline{g}(\underline{L})$$

The shapes of the curves of $L_1(t)$, $L_2(t)$, $L_3(t)$ are indeed strongly reminiscent of the elliptic functions. The analytical solution of this very interesting case has been achieved by Dr. Česlovas Masaitis of BRL(6). For our present descriptive purposes (and for blending with the more general four-dimensional case) the relationship of these functions to the elliptic functions is, unfortunately, complicated (they are generally square roots of the ratios of fourth-degree polynomials in elliptic functions, although there are some simplifications).

There are three pairs of "equilibrium" points (at which $\dot{\underline{L}} = 0$):

(A) $L_1 = L_2 = 0$, $L_3 = \pm L_{3A}$; viz., on axis 3.

(B) $L_1 = 0$, $\left[L_3(I_3^{-1} - K_2^{-1}) - L_6 I_3^{-1} \right] = 0$; viz., in the plane 2 - 3, at $L_2 = \pm L_{2B}$, $L_{3B} = L_6/(1 - I_3/K_2)$.

(C) $L_2 = 0$, $\left[L_3(I_3^{-1} - K_1^{-1}) - L_6 I_3^{-1} \right] = 0$; viz., in the plane 1 - 3, at $L_1 = \pm L_{1C}$, $L_{3C} = L_6/(1 - I_3/K_1)$.

The simplest way to determine which ones of these are saddle points (are unstable) and which are centers ("stable"), is to consider the kinetic energy ellipsoid, which in this case can be written as

$$L_1^2/K_1 + L_2^2/K_2 + (L_3 - L_6)^2/I_3 = 2T - L_6^2/I_6 = M, \text{ say, } > 0,$$

so that the center of this ellipsoid is raised* by L_6, and its semiaxes are $\sqrt{K_1 M}$, $\sqrt{K_2 M}$, $\sqrt{I_3 M}$. In our example this ellipsoid is roughly a thin horizontal "disc", with the major axis parallel to axis 2. It can be visualized from an inspection of Figures 8. When M is minimum this "disc" is wholly inside the sphere, touching it (in the vicinity of its major semiaxis $\sqrt{K_2 M}$) only at the points B; accordingly, these points,

* There is no loss of generality in reckoning L_6 positive.

in the vicinity of the saddle points of Figure 2, become centers. With a large M, the "disc" in effect merely slices off two (or just one) horizontal spherical segments; so that the points A are centers, too. The points C of tangency of the "disc" and sphere in the vicinity of the intermediate semiaxis $\sqrt{K_1 M}$ of the "disc" - and in the vicinity of what were the centers at axis 1 on Figure 2 - become saddle points.

As compared with Figure 2, Figures 8 call for an additional parameter because they no longer utilize the simplification $L_6/L_3 = I_6/(I_3 + I_6)$ of Figure 2.

The peculiarity of the behavior of this physical system is indeed due to this reversal; viz.,

the centers on axis 1 in Figure 2 become the saddle points C in Figures 8.

the saddle points on axis 2 in Figure 2 become the centers B in Figures 8; however,

the centers on axis 3 in Figure 2 remain the centers A in Figures 8.

10. VERY SMALL FRICTION

We have mentioned that when the friction is introduced and we revert to a fourth-order system, the points A (with the additional qualification that $\beta = 0$) remain equilibrium points. For the points B and C, however, the requirement that one or the other bracket of equations (c) is zero - i.e., that $L_{3B} = L_6/(1 - I_3/K_2)$ or $L_{3C} = L_6/(1 - I_3/K_1)$ - generally contradicts the requirement (for a point of equilibrium) that $\beta = 0$, i.e., that $L_3 = L_6(1 + I_3/I_6)$. Points B will remain points of equilibrium only in the case where

$$1/(1 - I_3/K_2) = 1 + I_3/I_6, \text{ or } K_3 = K_2$$

which is not the case in our example: $1/(1 - .1/1.1) \neq 1 + .1/2$, and which cannot be the case with both the assumption $I_4 = I_5$ and the requirement $I_2 \neq I_3$. Thus for $K_3 \neq K_2$ and $f \neq 0$, strictly speaking, equilibrium points B and C do not exist.

Similarly, we may no longer use the argument similar to that which we used in passing from the case of infinite friction to the case of fairly large friction, for the quantity M (which determines the size of the T-ellipsoid) does not necessarily have properties analogous to those of T ; viz., $\dot{M}$ as a function of t is not necessarily negative-definite.

Nevertheless, we may expect that with very small friction the trajectories will be <u>somewhat similar</u> to those of Figures 8. In this sense we shall speak of points B' as "quasi-centers" for the loops similar to those of Figures 8.

The stability of points B' so far (with our approach) could be determined only through numerical computation. It can be described (for our example only) as follows:

1. As far as the <u>loops about</u> the points B' are concerned, these quasi-centers are very stable asymptotically, in the sense that the motion damps to them very rapidly.

2. As far as the <u>location of</u> the points $\overset{\prime}{B}$ is concerned, these quasi-centers are unstable, with the axis 2 as the position of the unstable equilibrium of the points $\overset{\prime}{B}$. That is, if the loops around $\overset{\prime}{B}$ settle with the quasi-center $\overset{\prime}{B}$ at the axis 2 exactly, the vector $\underline{L}$ will remain at that point; but more generally, during these oscillations the point $\overset{\prime}{B}$ slips away from the axis 2, first extremely slowly, and then faster and faster.

3. The slippage of the points $\overset{\prime}{B}$ away from the axis 2 seems to proceed along a particular curve on our sphere of L^2.

4. This slippage of the points $\overset{\prime}{B}$ passes into a fast nutation about one of the centers A.

5. Nutations about the centers A appear asymptotically stable. However, the rate of damping of the nutations about these centers becomes extremely small - so that it becomes difficult to distinguish, with the differential analyzer, whether the centers A are truly asymptotically stable, or whether there might not exist some asymptotically-stable "small circle" about the poles A*.

These features can be seen on Figure 6 (f = .1) and Figure 7, (f = .02).

These empirical conclusions may, of course, be modified by a change in the law of friction employed.

* The stability of points A, and the fact that they are the only stable points have been recently proven by D. C. Lewis (7).

11. CRITICAL FRICTION

Now, of course, an increase in friction increases the damping. It is not difficult to find, by trials, a value of f (in our example this turns out to be approximately $f = .125$) with which the behavior of the system can be said to represent most sharply the border-line (or the blend) between the types of behavior represented by Figures 5 and 6. The behavior may perhaps be described as follows. Motion starting in the "polar zone" of our sketches passes at once into a damping nutation about axis 3. Motion starting in the "tropical" and "temperate" zones rapidly "drops" to the curve mentioned in item 3 above, and practically stops there for a while - and then proceeds, with increasing velocity, into a damping nutation about axis 3.

12. REMARKS AND DISCUSSION

We feel that a differential analyzer - provided it is adjusted very carefully (and/or provided with a number of auxiliary compensating circuits) - is an excellent tool for the rough "empirical" exploration of this sort. Its advantage over the digital machinery is the ease of access, of visualization, and of inspection of the "blends" between those types of behavior of the system which at first glance appear to be basically different from previous experience. Its shortcoming (in addition to the general shortcoming of the "empirical" mathematical approach) is a need for careful adjustments, and an apparent sensitivity of the machine to the type of circuitry chosen.

Our example was limited to the body 4, 5, 6 being a "thin disc" ($I_6 = 2I$). A few studies were done also with a "fat disc" ($1 < I_6/I < 2$). Surprisingly, the fat disc performs much better than the thin disc: the oscillations are slower, and the damping is far more pronounced. As the disc approaches a sphere ($I_6 \to I$), the vector $\underline{L}$ appears to start approaching the axis 3 asymptotically and without nutations. As the disc becomes a spindle ($I_6 < I$), however, the motion passes into a slow and quickly-damping nutation about axis 2. Thus this mechanism allows a "stabilization" of a spindle only in this sense: the spindle can be put eventually into a pure cartwheeling motion, in which the "inertial member" (the small body 1, 2, 3, which is long along axis 1) arranges itself perpendicular to the final axis of rotation (the axis 2), but in which the orientation of this axis with respect to the spindle (the principal body 4, 5, 6) is, unfortunately, aleatory.

A number of refinements of this system are possible. At this time we visualize only the following refinements:

(a) A more thorough inspection of the inequality

$$1/(1 - I_3/K_2) \neq 1 + I_3/I_6 ,$$

which as we have noticed, is the reason for the instability of the points B (and which represents an approach to the condition $K_3 = K_2$).

(b) An exploratory inspection of the case where the body 1, 2, 3 is the principal body ("a fat elliptical disc"), with the body 4, 5, 6 the small (symmetrical) "inertial member".

(c) An exploratory inspection of the possibility of reinforcing the inertial action of the assembly of the type (b) by "flywheel gearing".*

(d) An exploratory inspection of the effect of variation of the law of friction employed.

While this paper is limited to purely passive damping, the "empirical" approach based on these principles should eventually facilitate the inspection of more complicated mechanisms, which may possess springs, controls, power sources, and/or loss of mass of the satellite.

From the ballistic viewpoint, the most interesting feature of this device is its strong tendency to stick - temporarily - with the spin about the diameter of the disc (axis 2). It is this feature - as well as the possibility of constructing an equivalent of "negative friction" (viz, an energy source) - that seems to leave hope that an eventual "internal" stabilization of a spindle might be possible. However, that is a matter of "synthesis", i.e., of basic changes in the physical system considered here.

Serge J. Zaroodny
SERGE J. ZAROODNY

James W. Bradley
JAMES W. BRADLEY

* A crude experimental model of such a "flywheel gearing" device shows an excellent "nutation damping" performance. It is interesting to note that the equations of motion of such device are basically the same as considered here; only the expressions for the constant coefficients are different.

REFERENCES

1. Leimanis, E. "Some Recent Advances in the Dynamics of Rigid Bodies and Celestial Mechanics", Part I of Vol. II, "Dynamics and Non-Linear Mechanics", of "Surveys in Applied Mathematics", John Wiley & Sons, Inc. (1958).

2. Zaroodny, S. J. "An Exact Re-statement of the Equations of Motion of a Rigid Body for Use With Modern Computing Machinery", BRL Memorandum Report 856 (1954)

3. Murphy, C. H. "An Interesting Fourth-Order Differential System", Am. Math. Monthly, LXIV, 9, November 1957.

4. Zaroodny, S. J. "An Elementary Introduction to Elliptic Functions of a Real Variable Based on the Theory of Nutation of a Free Rigid Body", one of a series of ditto'd working papers on this problem, available at BDPB, EBL, BRL, APG, Md.; also, Am. Math. Monthly, Aug - Sep 1961.

5. Haseltine, W. R. "Passive Damping of Wobbling Satellites, Part I. General Stability Theory and First Example", NAVORD Report 6579 (1959).

6. Masaitis, Č. "On the Motion of Two Linked Rigid Bodies", BRL Report 1097, March, 1960.

7. Lewis, D. C. "Paper Without Title", a manuscript from RIAS, Baltimore, Maryland, January 1960.

LIST OF SYMBOLS

I moment of inertia of a rigid body

K sum of the two moments of inertia about the same axis

L angular momentum

T kinetic energy

t time; dot indicates d/dt

ω angular velocity

β $= \omega_6 - \omega_3$

f friction torque coefficient

F friction torque

M in the frictionless case, $2T - L_6^2/I_6$

Subscripts:

Numbers indicate an axis of the coordinate system

A,B,C indicate the equilibrium points in the frictionless case.

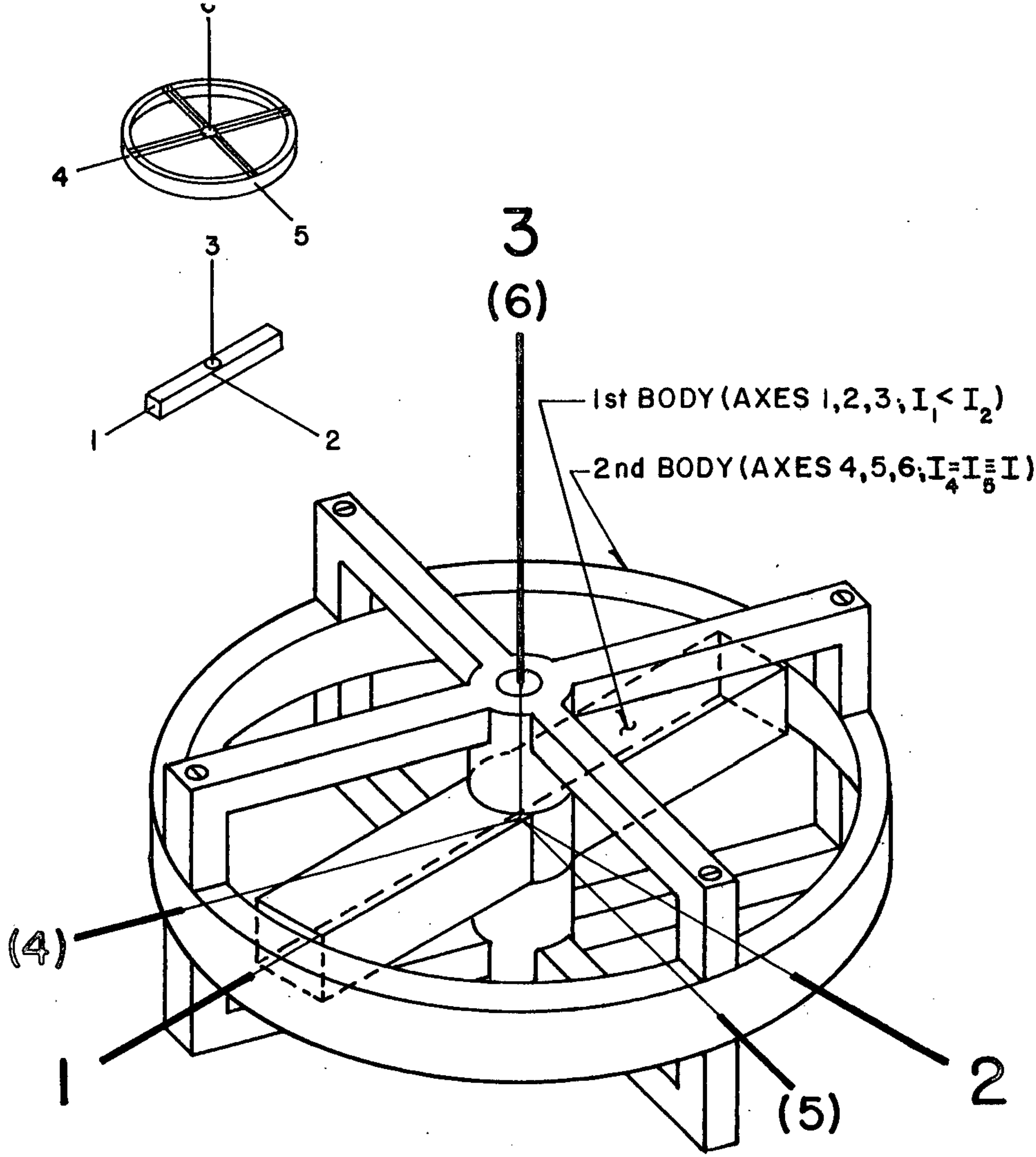

FIG. 1-SCHEMATIC ILLUSTRATION OF THE PRINCIPLES OF A NUTATION DAMPER, SHOWING THE COORDINATE SYSTEM IN WHICH, ON THE SUBSEQUENT FIGURES, THE MOTION OF THE TIP OF THE VECTOR $\underline{L}$ OF ANGULAR MOMENTUM IS GIVEN.

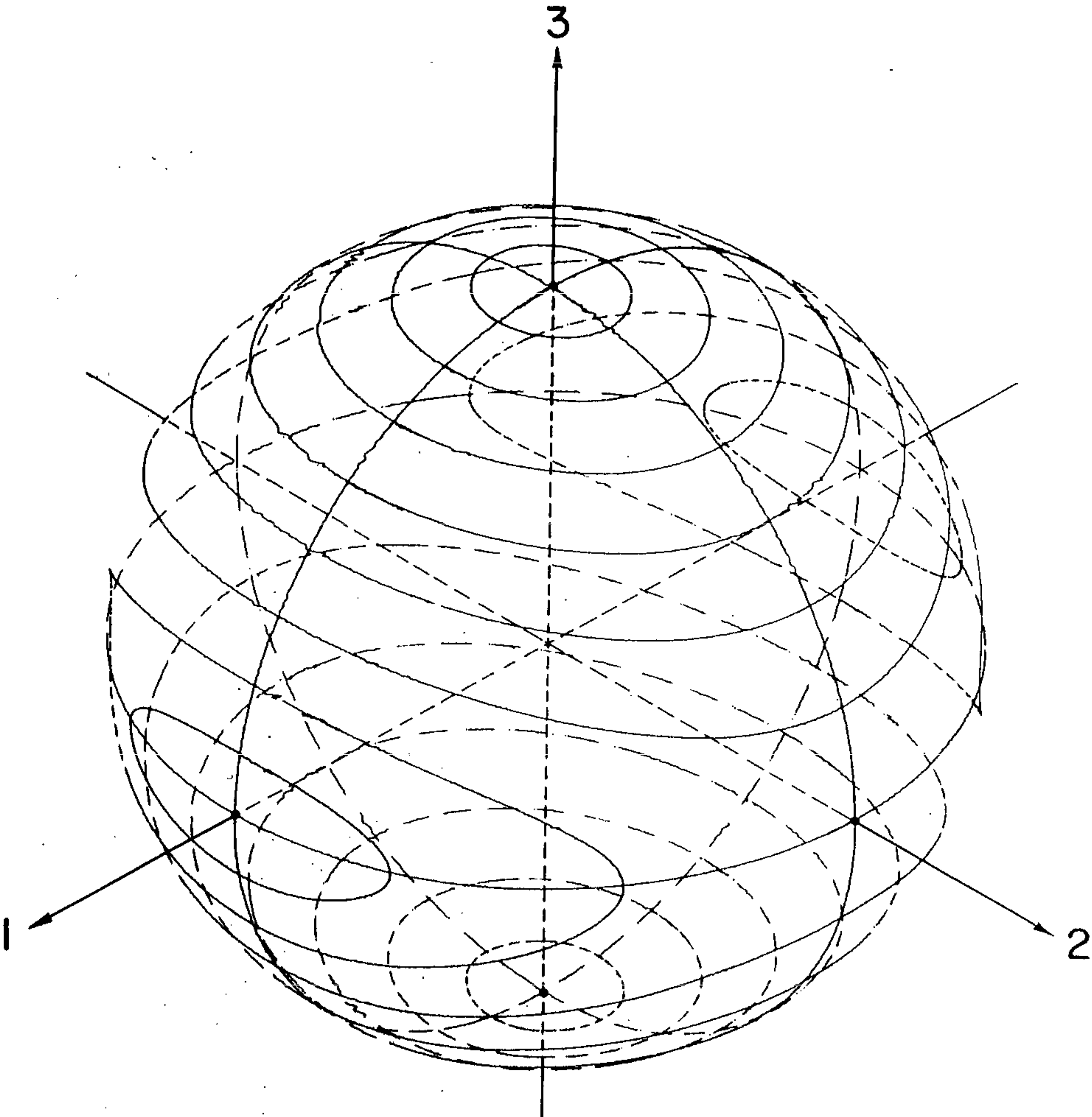

FIG. 2: POLHODES FOR INFINITE FRICTION ($f = \infty$).

SYSTEM IS A RIGID BODY. T IS CONSTANT.

NUTATIONS ABOUT AXES 1 AND 3 ARE STABLE, ABOUT AXIS 2 UNSTABLE.

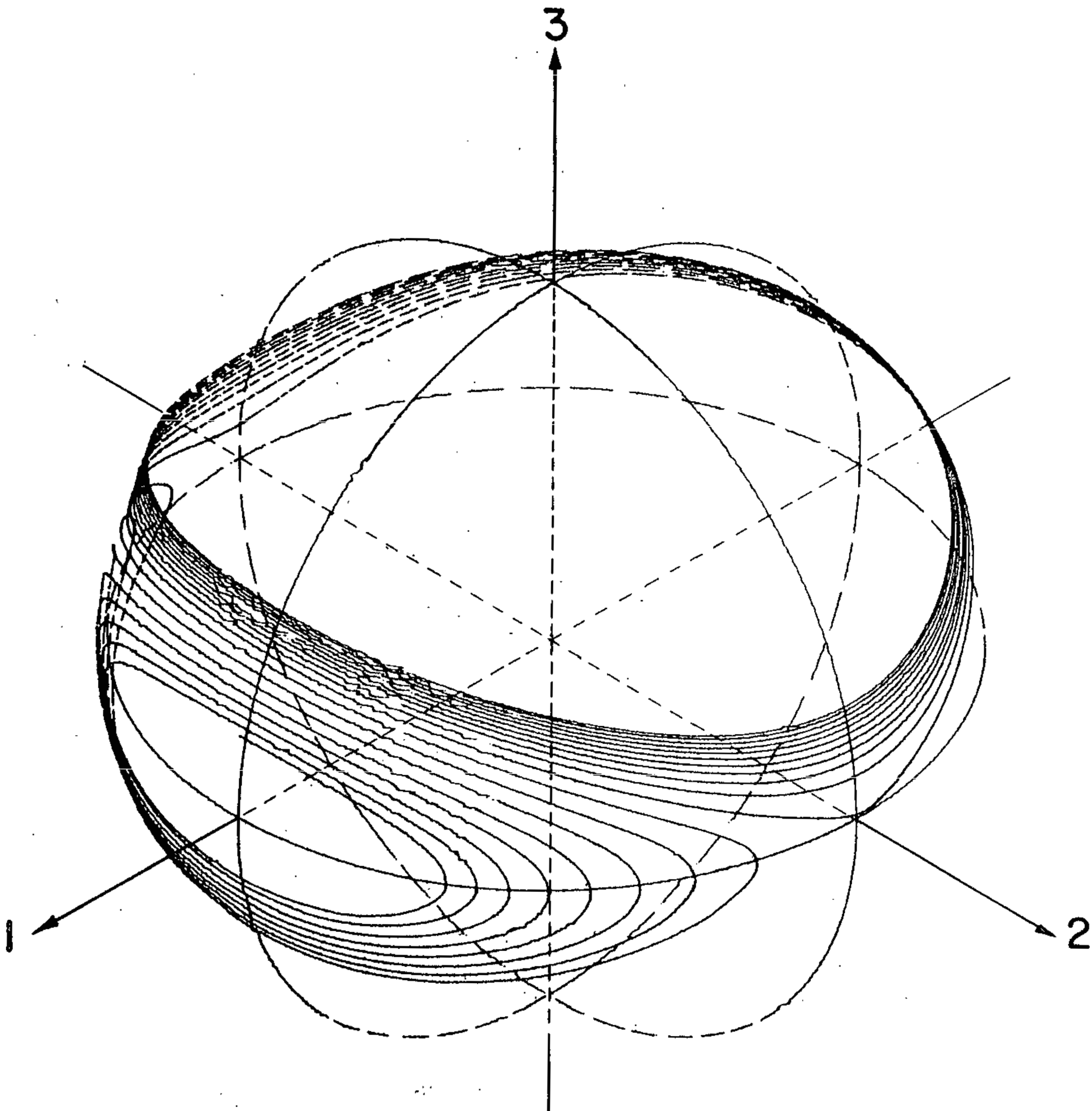

FIG. 3: POLHODES FOR FAIRLY LARGE FRICTION (f = 10)
NUTATIONS ABOUT AXIS 1 "UNDAMP" AND
PASS OVER TO DAMPED NUTATIONS ABOUT AXIS 3.

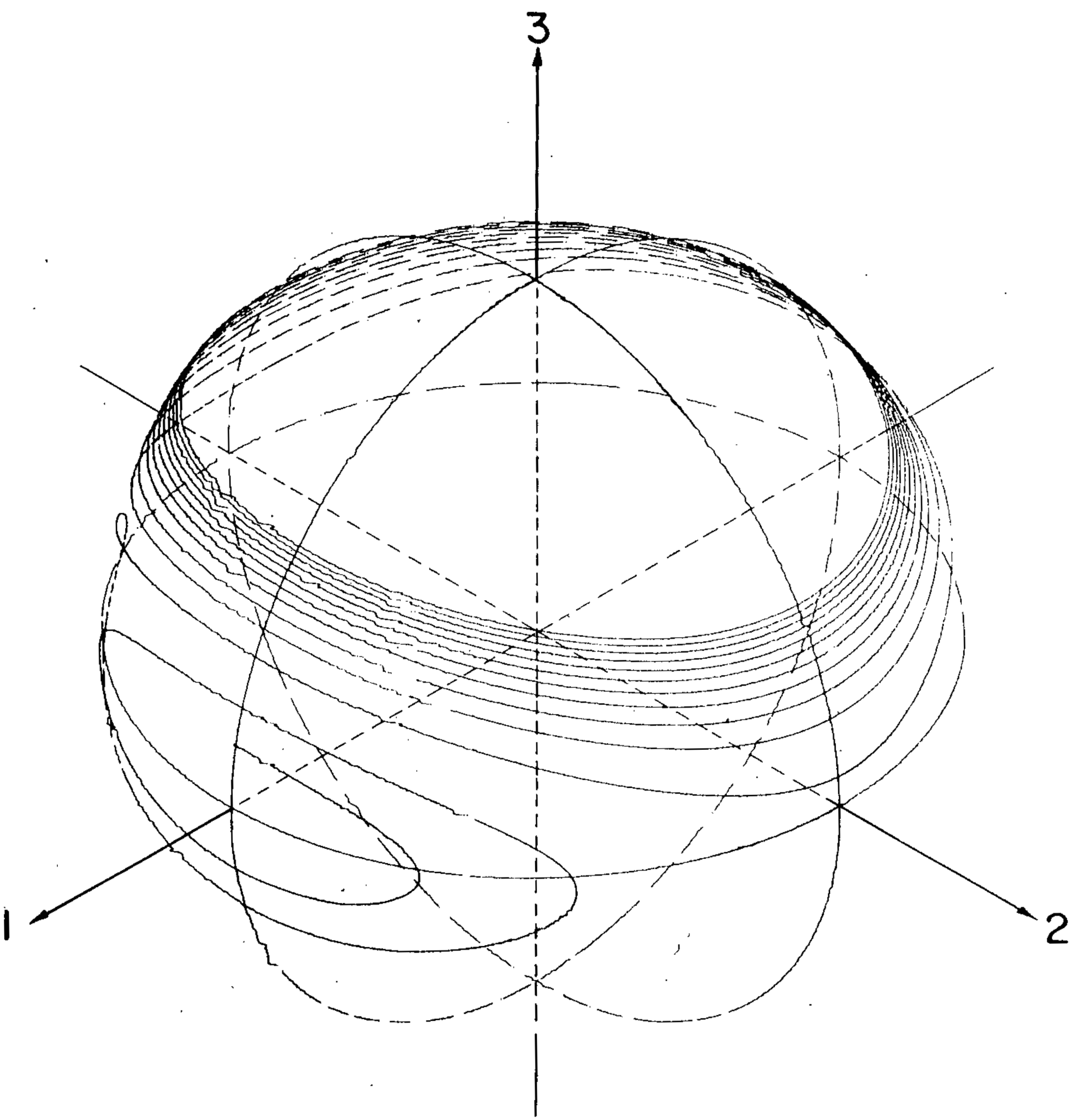

FIG. 4: POLHODES FOR MODERATE FRICTION (f = 2)

THE RATE OF "UNDAMPING" ABOUT AXIS 1
HAS INCREASED AND SO, TO A LESSER DEGREE, HAS THE
RATE OF DAMPING ABOUT AXIS 3.

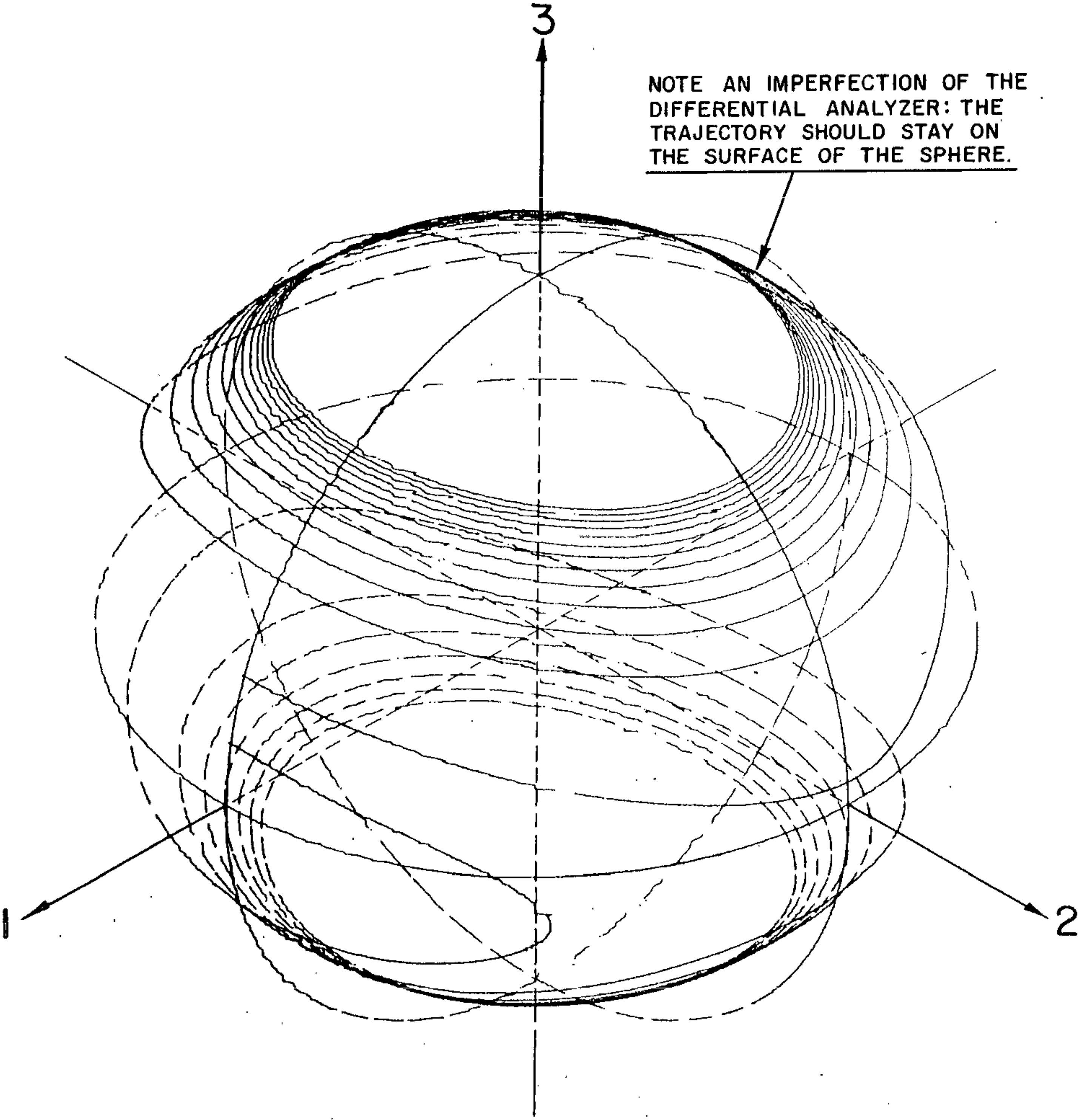

FIG. 5: POLHODES FOR MODERATE FRICTION (f = 0.5)

TWO CURVES ARE SHOWN, ONE SETTLING ABOUT THE POSITIVE AXIS 3, THE OTHER ABOUT THE NEGATIVE AXIS 3. THE "UNDAMPING" IS NOW VERY RAPID, BUT THE GAIN IN THE DAMPING RATE HAS DIMINISHED.

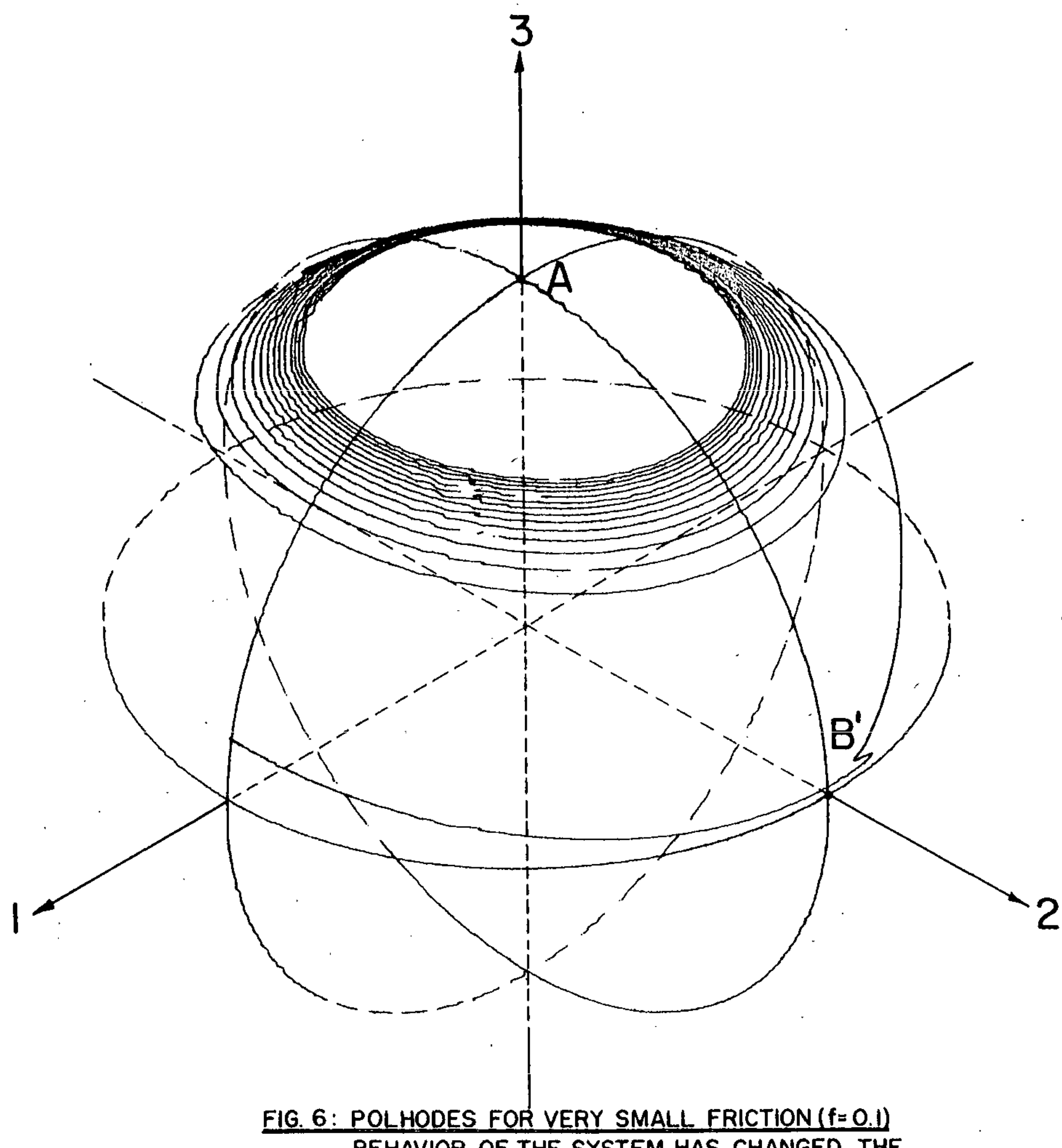

FIG. 6: POLHODES FOR VERY SMALL FRICTION (f= 0.1)
BEHAVIOR OF THE SYSTEM HAS CHANGED. THE VECTOR L TENDS TO LINGER ABOUT A "QUASI-CENTER" B' NEAR AXIS 2

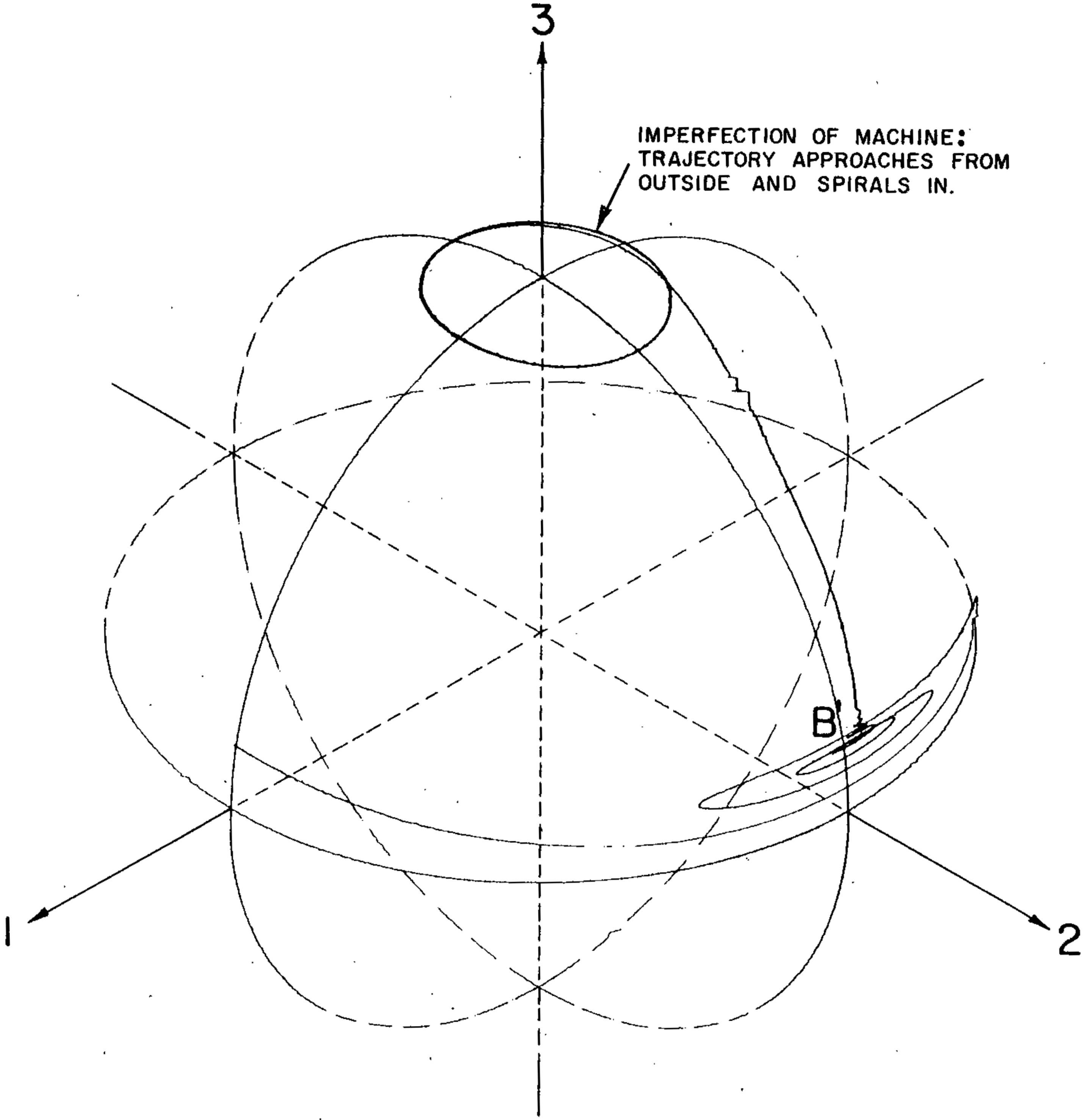

FIG. 7: POLHODES FOR VERY SMALL FRICTION (f=0.02)
THE "QUASI-CENTER", B', IS NOW CLEARLY DEFINED. AS L LOOPS THIS POINT, THE POINT SLIPS AWAY FROM AXIS 2, SLOWLY AT FIRST, BUT WITH INCREASING SPEED.

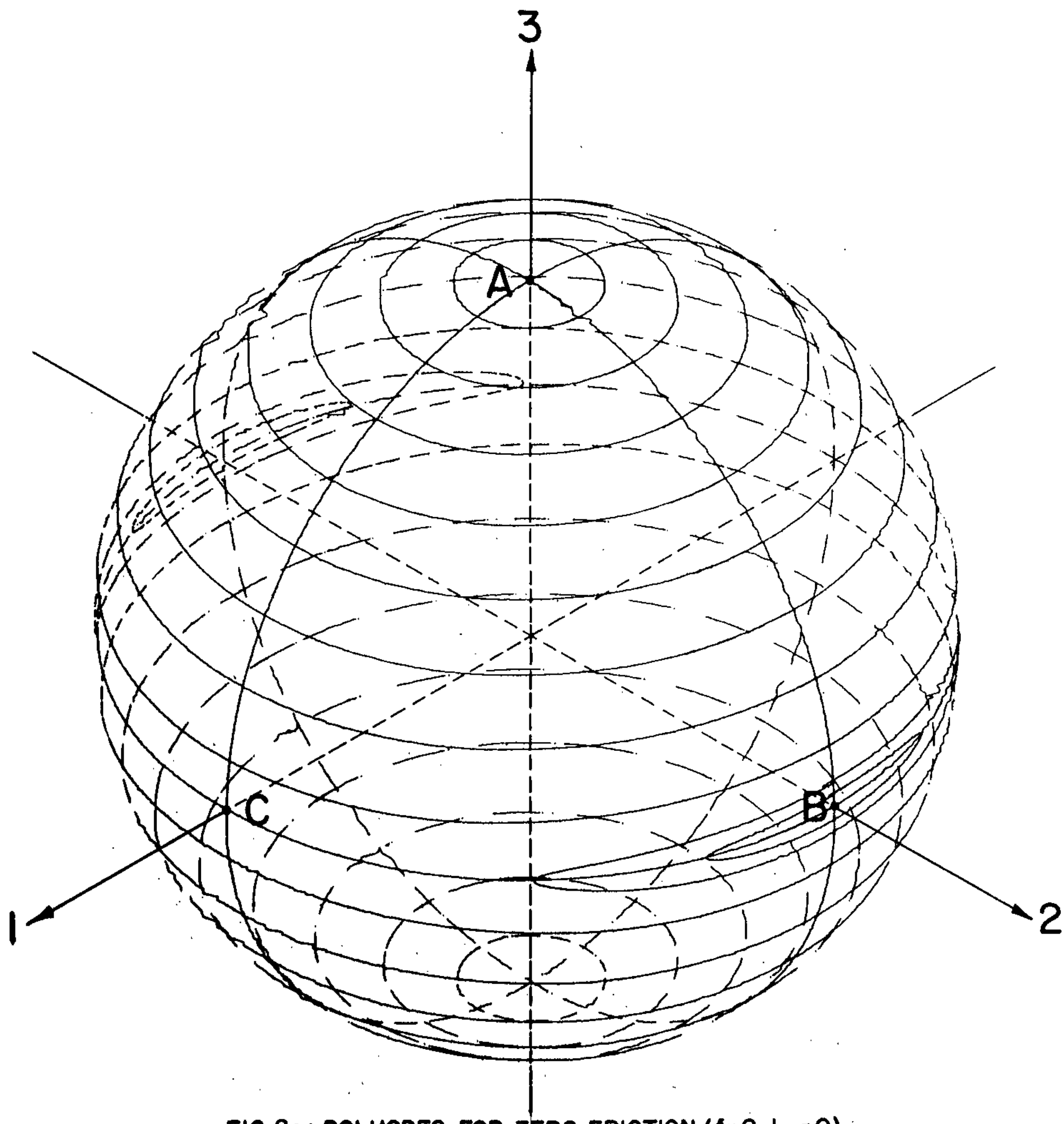

FIG. 8a: POLHODES FOR ZERO FRICTION ($f=0, L_e=0$)
SYSTEM IS A (HYPOTHETICAL) RIGID BODY. T IS CONSTANT. POINTS A AND B ARE CENTERS, POINT C IS A SADDLE.

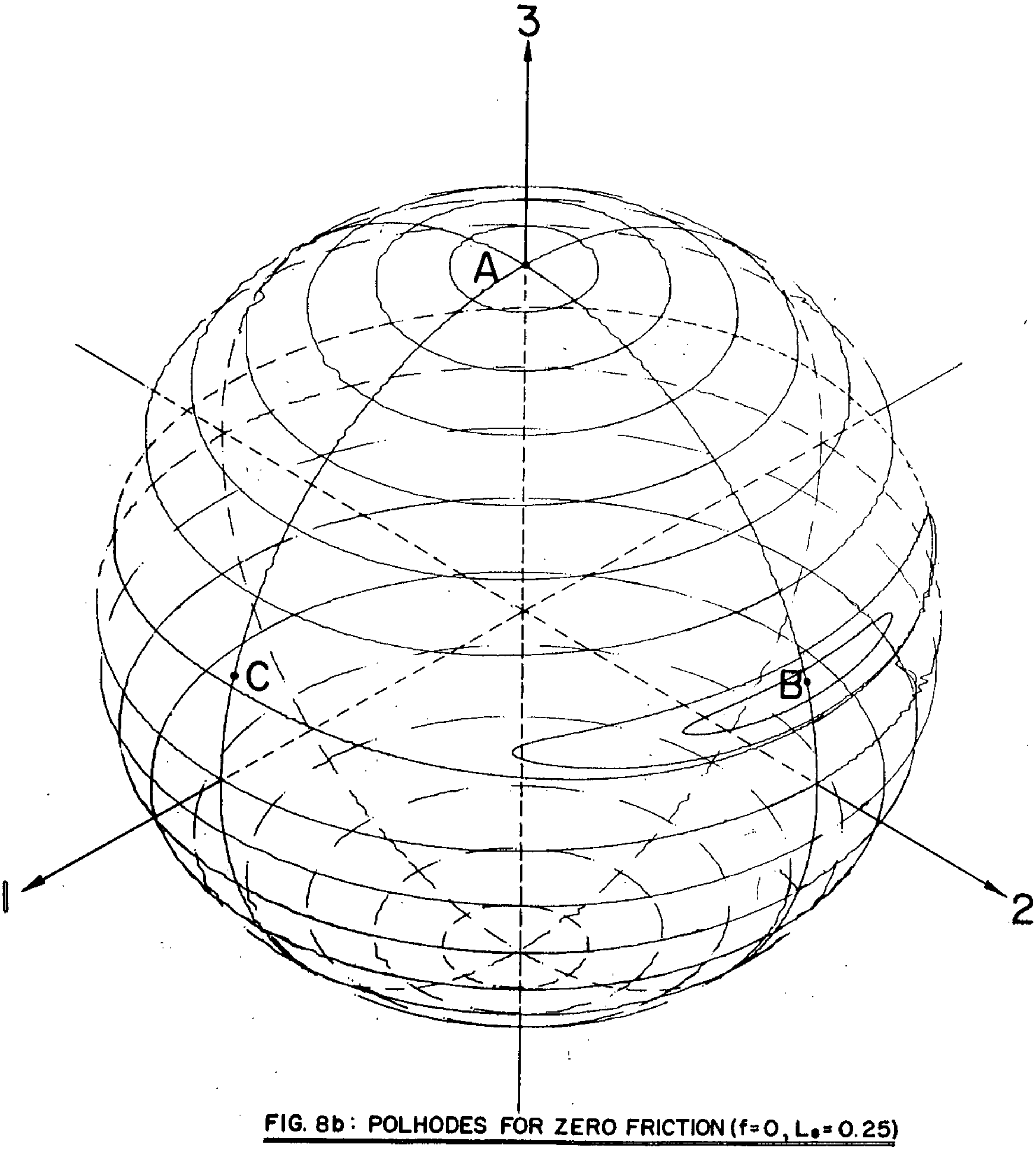

FIG. 8b: POLHODES FOR ZERO FRICTION ($f = 0$, $L_e = 0.25$)

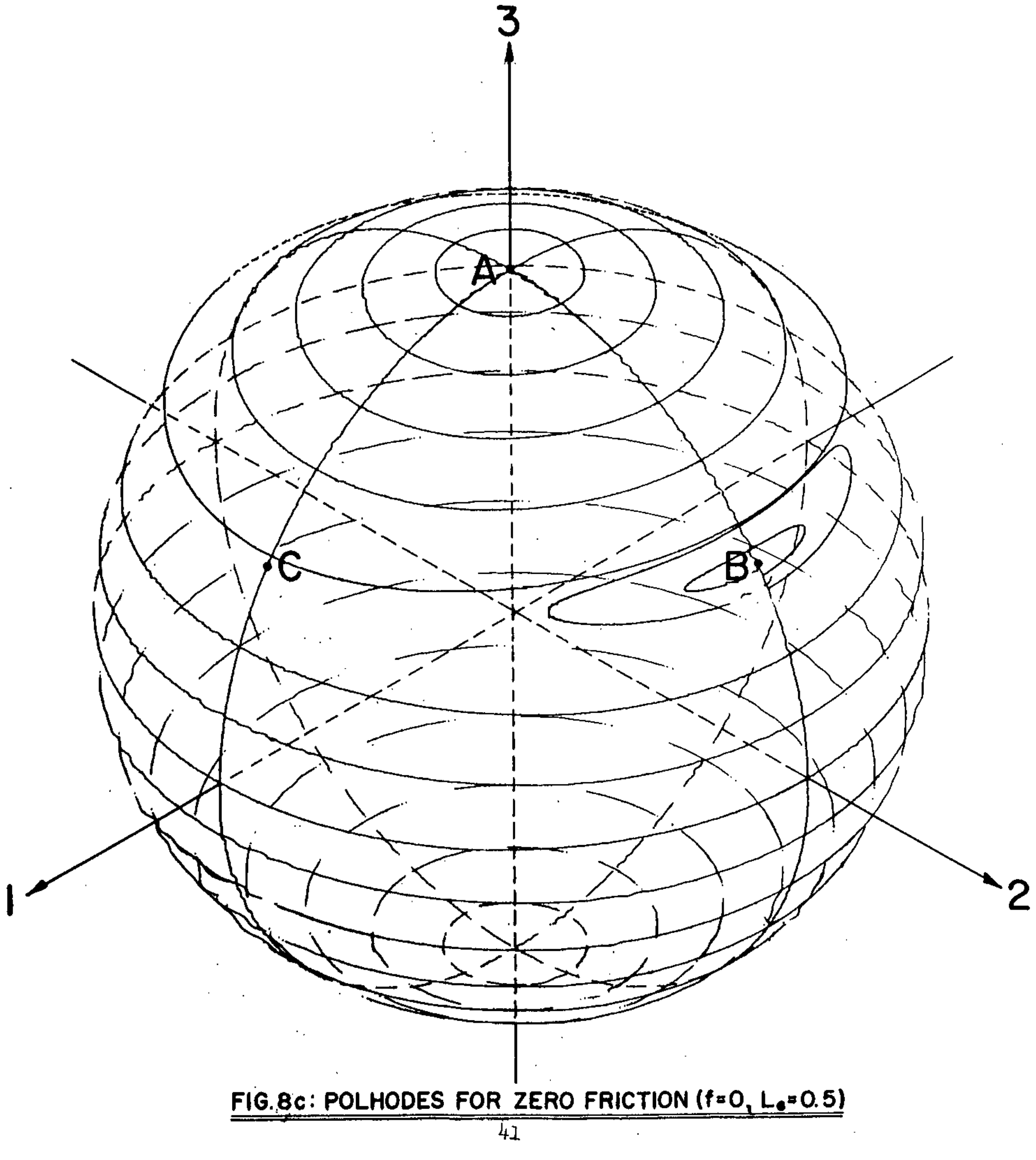

FIG. 8c: POLHODES FOR ZERO FRICTION ($f = 0$, $L_e = 0.5$)

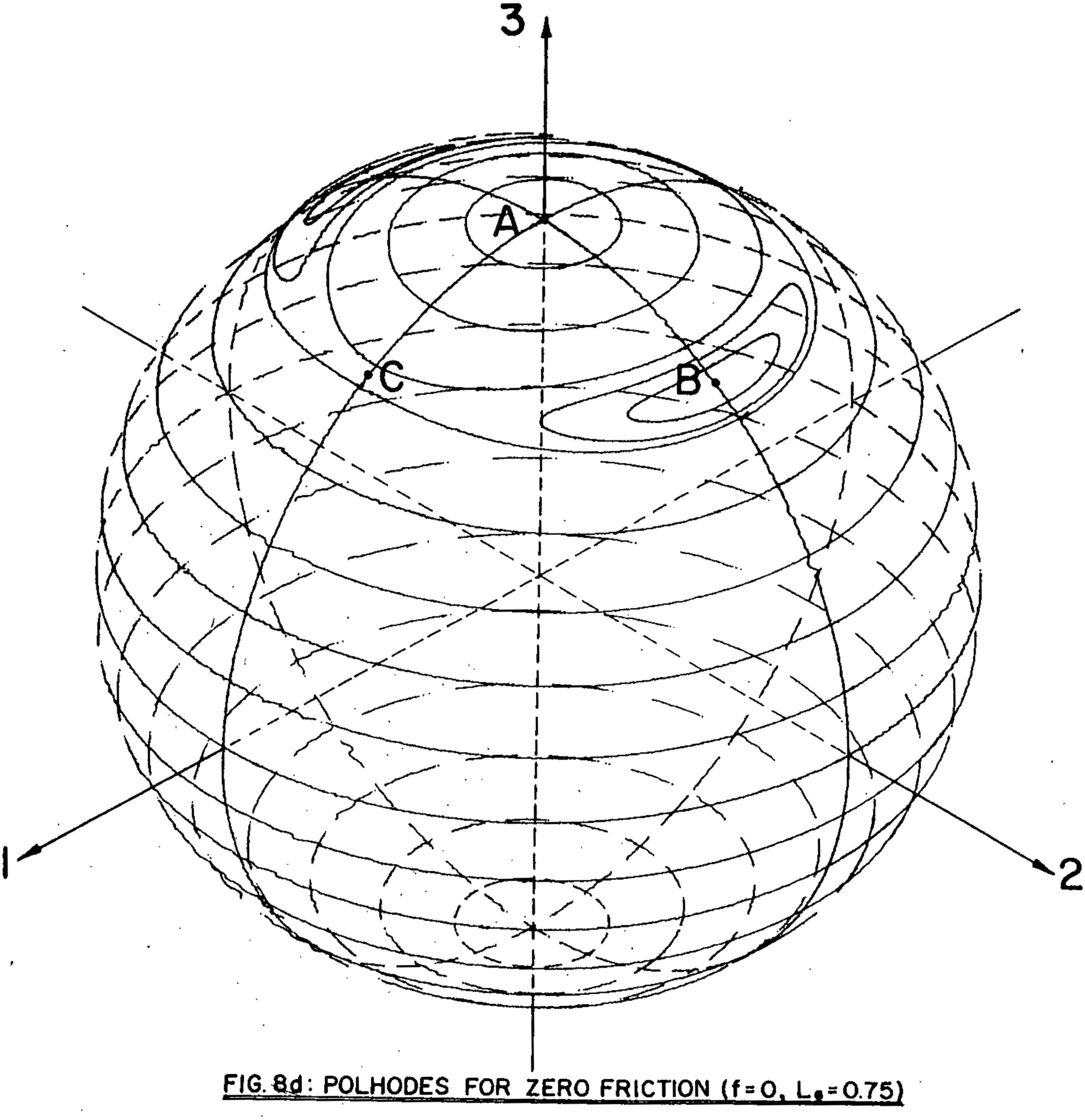

FIG. 8d: POLHODES FOR ZERO FRICTION ($f = 0$, $L_s = 0.75$)

DISTRIBUTION LIST

No. of Copies	Organization
1	Chief of Ordnance ATTN: ORDTB - Bal Sec Department of the Army Washington 25, D. C.
1	Commanding Officer Diamond Ordnance Fuze Laboratories ATTN: Technical Information Office, Branch 012 Washington 25, D. C.
10	Commander Armed Services Technical Information Agency ATTN: TIPCR Arlington Hall Station Arlington 12, Virginia
10	Commander British Army Staff British Defence Staff (W) ATTN: Reports Officer 3100 Massachusetts Avenue, N.W. Washington 8, D. C.
4	Defence Research Member Canadian Joint Staff 2450 Massachusetts Avenue, N.W. Washington 8, D. C.
1	Office of Technical Services Department of Commerce Washington 25, D. C.
3	Chief, Bureau of Naval Weapons ATTN: DIS-33 Navy Department Washington 25, D. C.
1	Chief, Bureau of Ships ATTN: Technical Library Code 335, Room 1532 Department of the Navy Washington, D. C.
1	Director U. S. Naval Research Laboratory Washington 25, D. C.

No. of Copies	Organization
2	Commander U. S. Naval Weapons Laboratory Dahlgren, Virginia Of Interest to: Dr. C. J. Cohen
2	Commander Naval Ordnance Laboratory ATTN: Mr. Nestingen Dr. May White Oak Silver Spring 19, Maryland
3	Commander Naval Ordnance Test Station ATTN: Technical Library Dr. W. R. Hazeltine China Lake, California
1	Commander Air Force Systems Command ATTN: Deputy for Development Andrews Air Force Base Washington 25, D. C.
1	Commander Aerospace Technical Intelligence Center ATTN: RRLM - Dr. Gertrude Blanch Wright-Patterson Air Force Base, Ohio
2	Commanding General Army Rocket and Guided Missile Agency ATTN: Technical Library, ORDXR-OTL Mr. Clifton Fitton Redstone Arsenal, Alabama
1	Commanding General Army Ballistic Missile Agency Redstone Arsenal, Alabama
1	Commanding Officer U. S. Army Research Office (Durham) ATTN: Dr. F. G. Dressel Box CM, Duke Station Durham, North Carolina

DISTRIBUTION LIST

No. of Copies	Organization
1	Army Research Office Arlington Hall Station Arlington, Virginia
2	Director National Aeronautics and Space Administration ATTN: Division of Research Information 1520 H Street, N.W. Washington 25, D. C.
1	Director Jet Propulsion Laboratory ATTN: Mr. Irl E. Newlan, Reports Group 4800 Oak Grove Drive Pasadena 3, California
1	Arthur D. Little, Inc. ATTN: Mr. W. A. Sawyer Acorn Park Cambridge 40, Massachusetts
2	Applied Physics Laboratory The Johns Hopkins University ATTN: Mr. George L. Seielstad 8621 Georgia Avenue Silver Spring, Maryland
1	Cornell Aeronautical Laboratory, Inc. ATTN: Miss Elma T. Evans Librarian 4455 Genesee Street Buffalo, New York
1	Parker Aircraft Company 5827 W. Century Boulevard Los Angeles 45, California
1	Flight Refueling, Inc. ATTN: Charles E. Comean Friendship International Airport Baltimore 3, Maryland
1	Professor D. J. Brouwer Yale University New Haven, Connecticut
1	Professor George F. Carrier Division of Engineering and Applied Physics Harvard University Cambridge 38, Massachusetts
1	Professor J. P. La Salle RIAS 7212 Bellona Avenue Baltimore 12, Maryland
1	Professor R. Langer USA MCR University of Wisconsin Madison, Wisconsin
1	Professor C. E. Langenhop Iowa State College Ames, Iowa
1	Dr. D. C. Lewis RIAS 7212 Bellona Avenue Baltimore 12, Maryland
1	Professor W. S. Loud University of Minnesota Minneapolis 14, Minnesota
1	Dr. C. V. L. Smith National Aeronautics and Space Administration Data Reduction Division 1520 H Street, N.W. Washington 25, D. C.
1	Professor J. J. Stoker New York University Institute of Mathematics and Mechanics New York 53, New York
1	Professor C. A. Truesdell Indiana University Bloomington, Indiana

AD ______________ Accession No. ______________

UNCLASSIFIED
Nutation Damping - Satellites
Satellites - Nutation Damping

Ballistic Research Laboratories, APG
NUTATION DAMPER - A SIMPLE TWO-BODY GYROSCOPIC SYSTEM
S. J. Zaroodny and J. W. Bradley

BRL Report No. 1128 April 1961

DA Proj No. 503-03-001, OMSC No. 5010.11.814
UNCLASSIFIED Report

The behavior of an assembly of two rigid bodies - a device suggested as a "nutation damper" for observation satellites - is studied. For the present empirical approach, one of the bodies is chosen to be a "thin rod", the other a "thin disc"; solutions of the equations are inspected for various values of the friction parameter.

AD ______________ Accession No. ______________

UNCLASSIFIED
Nutation Damping - Satellites
Satellites - Nutation Damping

Ballistic Research Laboratories, APG
NUTATION DAMPER - A SIMPLE TWO-BODY GYROSCOPIC SYSTEM
S. J. Zaroodny and J. W. Bradley

BRL Report No. 1128 April 1961

DA Proj No. 503-03-001, OMSC No. 5010.11.814
UNCLASSIFIED Report

The behavior of an assembly of two rigid bodies - a device suggested as a "nutation damper" for observation satellites - is studied. For the present empirical approach, one of the bodies is chosen to be a "thin rod", the other a "thin disc"; solutions of the equations are inspected for various values of the friction parameter.

AD ______________ Accession No. ______________

UNCLASSIFIED
Nutation Damping - Satellites
Satellites - Nutation Damping

Ballistic Research Laboratories, APG
NUTATION DAMPER - A SIMPLE TWO-BODY GYROSCOPIC SYSTEM
S. J. Zaroodny and J. W. Bradley
BRL Report No. 1128 April 1961

DA Proj No. 503-03-001, OMSC No. 5010.11.814
UNCLASSIFIED Report

The behavior of an assembly of two rigid bodies - a device suggested as a "nutation damper" for observation satellites - is studied. For the present empirical approach, one of the bodies is chosen to be a "thin rod", the other a "thin disc"; solutions of the equations are inspected for various values of the friction parameter.

AD ______________ Accession No. ______________

UNCLASSIFIED
Nutation Damping - Satellites
Satellites - Nutation Damping

Ballistic Research Laboratories, APG
NUTATION DAMPER - A SIMPLE TWO-BODY GYROSCOPIC SYSTEM
S. J. Zaroodny and J. W. Bradley
BRL Report No. 1128 April 1961

DA Proj No. 503-03-001, OMSC No. 5010.11.814
UNCLASSIFIED Report

The behavior of an assembly of two rigid bodies - a device suggested as a "nutation damper" for observation satellites - is studied. For the present empirical approach, one of the bodies is chosen to be a "thin rod", the other a "thin disc"; solutions of the equations are inspected for various values of the friction parameter.